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On the Kostant multiplicity formula

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Dedicated to I.M. Gelfand
on his 75th birthday

Abstract. *The Kostant multiplicity formula is a recipe for computing the weight multiplicities of an irreducible representatation of a compact semi-simple Lie group. We describe here a generalization of Kostant's formula: Suppose* τ *is a Hamiltonian action of a compact Lie group on a compact symplectic manifold. For an appropriate «quantization»,* τ^Q, *of* τ *the weight multiplicaties of* τ^Q *are given by a formula similar to Konstant's. There is also an asymptotic version of this formula which gives a recipe for computing the Duistermaat Heckman polynomials associated with* τ.

SECTION 1. INTRODUCTION

Let G be a torus and ρ a finite dimensional complex representation of G. One of the basic questions that can be asked about ρ is its decomposition into irreducibles. Since all irreducibles of G are one dimensional, and given by integral weights, α, on the Lie algebra of G, the question is how to describe the multiplicity of any given weight α. If G is the maximal torus of a compact semi-simple group K, and ρ_λ is the restriction to

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G of an irreducible representation of K with maximal weight λ, then the answer to this question is given by the celebrated Kostant multiplicity formula [K] which says that the multiplicity $m(\alpha, \rho_\lambda)$ is given by

$$m(\alpha, \rho_\lambda) = \sum(-1)^w N(w(\lambda + \alpha) - (\alpha + \delta)). \tag{1.1}$$

In this expression δ is one half the sum of the positive roots and the sum ranges over all w in the Weyl group. The function N is the Kostant partition function: $N(\upsilon)$ is the number of solutions in non-negative integers $(n_1, \ldots, n_k)$ of the equation

$$\upsilon = n_1\alpha_1 + \ldots + n_k\alpha_k \tag{1.2}$$

where the α_j are the positive roots. Although (1.1) is an explicit formula, it is extremely difficult to evaluate, since all partition functions, including the Kostant partition function, are notoriously diffucult to evaluate. Furthermore, there are some miraculous cancellations which occur in the Kostant multiplicity formula. For example, it is known that the set of all α for which $m(\alpha, \rho_\lambda) \neq 0$ is the convex hull of the points $\{w\lambda\}$ as w ranges over the Weyl group. In particular, $m(\alpha, \rho_\lambda) = 0$ if α does not lie in this convex hull, a fact that is not at all obvious from the right hand side of (1.1). The Kostant multiplicity formula can be derived from the Weyl character formula, cf. [J]. A modern discussion of this whole circle of ideas from the algebraic point of view can be found in the papers [B-G-G]. The Weyl character formula can be derived geometrically from a combination of the Bott-Borel-Weil theorem and the Atiyah-Bott fixed theorem, cf. [A-B]. In the more general setting of the Atiyah-Bott theorem we are given a holomorphic action of G on a holomorphic line bundle L over a compact Kaehler manifold M with isolated fixed points. If a ξ in the Lie algebra of G is such that $\exp(\xi)$ is regular at all the fixed points, then the Atiyah-Bott theorem expresses the Lefschetz number of $\exp(\xi)$ in terms of a sum over the fixed points, p, of an expression involving $\exp(\xi)$ and p. The regularity condition means the following : Let p be a fixed point of G. Then there is a linear action of G on the tangent space T_p and this action will have certain weights, $\alpha_{p,i}$. None of these can be the zero weight, for a zero weight would imply a (real) two dimensional subspace of T_p consisting of points left fixed by all of G and hence (by the exponential map relative to the Kaehler metric or any G invariant Riemann metric) a two dimensional manifold of fixed points passing through p, contradicting the assumption of isolated fixed points. Then the regularity condition is

$$\exp\, i\,\langle\alpha_{j,p}, \xi\rangle \neq 1$$

or

$$\langle\alpha_{j,p}, \xi\rangle \notin 2\pi\mathbb{Z} \quad \text{for all } p \text{ and } j. \tag{1.3}$$

Condition (1.3) clearly involves avoiding a countable number of hyperplanes. Let $\Phi : M \to g^*$ be the moment map associated to the action of G on M, where g^* is the dual space of the Lie algebra, g, of G. Then the Atiyah-Bott formula asserts that the Lefshetz number

$$L(\exp \xi) = \sum_p \frac{\exp i(\Phi(p), \xi)}{\prod_{k=1}^{N}(1 - \exp i\langle \alpha_{k,p}, \xi\rangle)} \tag{1.4}$$

Notice that condition (1.3) guarantees that the denominators of the summands in (1.4) do not vanish. A fixed point q is called a *vertex* if there exists a $\xi \in g$ such that

$$\langle \alpha_{j,q}, \xi\rangle > 0 \quad \text{for all } j. \tag{1.5}$$

The convexity theorem [A] and [G-S1] (see also [G-S3]) asserts that the image of the moment map is the convex hull of the set $\{\Phi(p)\}$ as p ranges over all the vertices. (The remaining fixed points if any will be called the interior fixed points. It is a property of actions coming from coadjoint orbits when G is the maximal torus that there are no interior fixed points.) Suppose we fix a vertex, q, and then choose a ξ so that (1.5) and (1.3) hold. This choice of ξ will amount to the analogue, in the general case, of the choice of a positive Weyl chamber implicit in the Kostant multiplicity formula. In particular, at every other fixed point p we have

$$\langle \alpha_{j,p}, \xi\rangle \neq 0. \tag{1.6}$$

Let

$$\epsilon_{j,p} = \operatorname{sign}(\langle \alpha_{j,p}, \xi\rangle), \quad \alpha^w_{jp} = \epsilon_{p,j}\alpha_{jp} \tag{1.7}$$

so that now for all p and j we have

$$\langle \alpha^w_{j,p}, \xi\rangle > 0. \tag{1.8}$$

We will now write down a generalization of Kostant's multiplicity formula. We will derive it from the Atiyah- Bott formula in section 4. Define N_p, the «partition function at p» by taking $N_p(v)$ to be the number of solutions in integers of the equation

$$v = n_1\alpha^w_{1,p} + \dots + n_d\alpha^w_{d,p}. \tag{1.9}$$

Let

$$(-1)^p = \prod_j \epsilon_{p,j}, \tag{1.10}$$

$$\delta_p = \frac{1}{2} \sum \alpha_{j,p}, \tag{1.11}$$

and

$$\delta_p^w = \frac{1}{2} \sum \alpha_{j,p}^w. \tag{1.12}$$

Let $\ell(\alpha, L)$ denote the Leftschetz number of α, that is the alternating sum of the multiplicity of α in the cohomology of the Doulbeault complex associated to L. Then our generalization of the Kostant multiplicity formula is

$$\ell(\alpha, L) = \sum_p (-1)^p N_p(\alpha + \delta_p^w - (\Phi(p) + \delta_p)). \tag{1.13}$$

(Notice that the quantities $N_p(v)$, (1.7), (1.10) and (1.12) and hence formula (1.13) all depend on the choice of q. Notice also that the sum in (1.13) extends over all fixed points, not just vertices.) As we have indicated, the evaluation of the partition function is a difficult business, and hence it is useful to have an «asymptotic» approximation to the multiplicity formulas (1.1) and (1.13). Let us explain what we mean by «asymptotic» in the context of the Kostant multiplicity formula (1.1). We can think of the multiplicity as a measure $\mu(\lambda)$, on g^*, where the measure $\mu(\lambda)$ is the sum over α of $m(\alpha, \rho_\lambda)$ times the delta function at α. So $\mu(\lambda)$, is a discrete measure supported at «lattice points» in the convex hull of $\{w\lambda\}$. Suppose we replace λ by $k\lambda$ where k is some large integer. As is well known, the representation $\rho_{k\lambda}$ is the highest weight component of the representation on the $k-th$ tensor power, $V^{\otimes k}$, where V is the underlying vector space of the representation ρ_λ. Its multiplicity measure, $\mu(k\lambda)$ will be a discrete measure supported on the convex hull of $\{kw\lambda\}$. Let $A_c : g^* \to g^*$ denote multiplication by the scalar c, so that

$$A_{(1/k)}\,[\text{convex hull of } \{kw\lambda\}] = \text{convex hull of } \{w\,\lambda\}.$$

So the push forward measure, $A_{(1/k)*}\mu(k\lambda)$, is a discrete measure supported on the «$(1/k)$ lattice points» in the convex hull of $\{w\lambda\}$. It is known [G-S2] that ,

$$\lim_{k\to\infty} A_{(1/k)*}\mu(k\lambda) = v_\Phi, \tag{1.14}$$

where

$$v_\Phi = \Phi_*\,(\text{Liouville meas. on } M), \quad \Phi : M \to g^* \text{ the moment map}, \tag{1.15}$$

and where M is the coadjoint orbit of K passing through λ. So v_Φ is our «asymptotic approximation» for the Kostant multiplicity measure. Now the Duistermaat-Heckman theorem [D-H] says that for any Hamiltonian G action, v_Φ is absolutely continuous with respect to Lebesgue measure on g^* and its Radon-Nikodym derivative is a piecewise polynomial function. More precisely, it says the following: Let $\Delta = \Phi(M)$ be the convex polytope which is the image of the moment map. Then Δ is a union of subpolytopes

$$\Delta = \Delta_1 \cup \ldots \Delta_i \cup \ldots \tag{1.16}$$

such that the interiors of the Δ_i consist of the regular values of Φ and

$$v_\Phi = f_i \times \text{(Lebesgue measure)} \quad \text{on } \Delta_i, \tag{1.17}$$

where the f_i are polynomials, which we shall call the Duistermaat-Heckman polynomials (or $D - H$ polynomials for short). So the problem of the asymptotics of the Kostant multiplicity formula (and its generalization) consists of evaluating the $D - H$ polynomials. One such scheme is due to Heckman [H] for the case of coadjoint orbits and motivates much of this paper: In the Kostant multiplicity formula (1.1) replace the Kostant multiplicity function N by (a suitable constant times) the volume of the polytope

$$v = x_1\alpha_1 + \ldots + x_d\alpha_d, \; x_1 \geq 0, \ldots x_d \geq 0. \tag{1.18}$$

To understand why such an expression should be relevant to the push forward of Liouville measure under the moment map, let us consider the situation near a vertex, p. By the equivariant Darboux theorem (see for example [G-S3]) the action of G near p (or near any fixed point for that matter) is equivalent to the linear action of G near the origin in the tangent space, T_p. For linear actions, the moment map is essentially given by a projection (we shall review this fact in section 2) and the $D - H$ polynomial can be evaluated by elementary geometry to have the form of a constant times the volume of a polytope such as (1.18). Indeed, let $\mathbb{R}^d_+$ denote the «positive d-tant» in $\mathbb{R}^d$ consisting of points with all coordinates positive, and let ds be the measure on $\mathbb{R}^d$ which is Lebesgue measure on $\mathbb{R}^d_+$ and vanishes outside $\mathbb{R}^d_+$. Let

$$L_p : \mathbb{R}^d_+ \to g^*$$

be the map defined by

$$L_p(s_1, \ldots, s_d) = \Phi(p) + s_1\alpha_{1,p} + \ldots + s_d\alpha_{d,p}. \tag{1.19}$$

Then if p is a vertex,

$$v_\Phi = L_{p*}ds \quad \text{near } \Phi(p), \tag{1.20}$$

and the right hand side of (1.20) is easily seen to have the form of a polynomial times Lebesgue measure, where the polynomial is given by a constant times the volume of a region such as (1.18). This will be explained in more detail in sections 2 and 3. Now suppose that we fix some vertex, q, and some $\xi \in g$ satisfying (1.5). Let us «renormalize» the maps L_p at all other fixed points by defining

$$L_p^w(s_1, s_d) = \Phi(p) + s_1\alpha_{1p}^w + \ldots + s_d\alpha_{dp}^w, \tag{1.21}$$

and set

$$\upsilon_p = L_{p*}^w \mathrm{d}\, s. \tag{1.22}$$

Then our generalization of Heckman's formula, to be proved in section 3 is

$$\upsilon_\Phi = \sum_p (-1)^p \upsilon_p. \tag{1.23}$$

Notice that once again some marvelous cancellations occur in (1.23) For example, υ_Φ vanishes outside Δ and has the much simpler form (1.20) on any region, Δ_i, abutting the image, $\Phi(p)$, of a vertex. So instead of using the closed formula (1.23), a more effective way of computing the $D-H$ polynomial in many cases is algorithmic: start with the known form of f_i in some subpolytope Δ_i and see how f_i changes when we cross a wall and move from Δ_i to an adjoining subpolytope. We shall provide formulas for the jumps across walls in section 5.

SECTION 2 COMPUTING THE $D-H$ POLYNOMIALS: THE LINEAR CASE

In this section we show how to compute the $D-H$ polynomials f_i on regions, Δ_i, adjacent to the exterior vertices of Δ. For such regions the action of G can be assumed to be a linear action, by the equivariant Darboux theorem, cf [G-S3]; so we will start by investigating in detail the linear case. For simplicity let G be the standard n-dimensional torus, i.e.

$$G = \mathbf{R}^n/\mathbb{Z}^n$$

Let V be a symplectic vector space, and ρ a representation of G on V. One can decompose V into a sum of invariant subspaces corresponding to the weights that occur in ρ :

$$V = \sum V^{\alpha_i}, \alpha_i \in (\mathbb{Z}^n)^* \tag{2.1}$$

We will assume that zero is not a weight of ρ, and hence that each V^{α} is a two-dimensional symplectic subspace of V (i.e. a one-dimensional complex subspace. Let $z_1, \ldots, z_N$ be a system of complex coordinates on V compatible with the decomposition, (2.1). Then the moment map associated with ρ is the map.

$$\Phi : V \to (\mathbf{R}^n)^*, \Phi(z_1, \ldots, z_N) = |z_1|^2 \alpha_1 + \ldots + |z_N|^2 \alpha_N \tag{2.2}$$

and the image is the convex conic polytope

$$\Delta = \{s_1\alpha_1 + \ldots + s_N\alpha_N, s_1, \ldots, s_N \geq 0\}. \tag{2.3}$$

The following is immediate:

LEMMA 2.1. *The following are equivalent:* Φ *is proper* $\Leftrightarrow$ Δ *is properly contained in a half-space, i.e. there exists a* $\xi \in \mathbf{R}^n$ *such that* $\langle \alpha_i, \xi \rangle > 0$ *for all* i. ∎

Lets assume from now on that this is the case. Then the push-forward of the symplectic measure on V with respect to Φ, i.e. the measure

$$v = v_{\Phi} = \Phi^* \left(\left(\frac{1}{\pi i} \right)^N \mathrm{d}\, z_1 \wedge \mathrm{d}\, \bar{z}_1 \wedge \ldots \wedge \mathrm{d}\, z_N \wedge \mathrm{d}\, \bar{z}_N \right), \tag{2.4}$$

is well-defined. We will show that one can compute it by elementary methods. In fact we will give below three equivalent descriptions of it, each one being useful for certain purposes. To begin with suppose $N = 1$. Set $\alpha = \alpha_1$. Then Δ is the single ray, $\{t\alpha, t > 0\}$. Let ι_{α} be the map of the half-line, $0 < t < \infty$, into $(\mathbf{R}^n)^*$ sending t to $t\alpha$.

LEMMA 2.2. v_{Φ} *is the push-forward with respect to* ι_{α} *of the Lebesgue measure,* $\mathrm{d}\, t$.

Proof. We want to show

$$\Phi * \frac{\mathrm{d}\, z \wedge \mathrm{d}\, \bar{z}}{\pi i} = (\iota_{\alpha})_* \mathrm{d}\, t \tag{2.5}$$

The LHS of (2.5), evaluated on a smooth, compactly supported function, f, is

$$\int \Phi^* f(z, \bar{z}) \frac{\mathrm{d}\, z \wedge \mathrm{d}\, \bar{z}}{\pi i} = \int f(|z|^2 \alpha) \frac{\mathrm{d}\, z \wedge \mathrm{d}\, \bar{z}}{\pi i}$$

The second expression can be written in polar coordinates, $z = re^{i\theta}$, as

$$2 \int_0^{\infty} f(r^2 \alpha) r \mathrm{d}\, r = \int_0^{\infty} f(t\alpha) \mathrm{d}\, t$$

and the expression on the right is the RHS of (2.5) evaluated on f. Q.E.D. ∎

Now consider the general case. By (2.1) the representation, ρ, is the product of the representations of G on the one dimensional subspaces, V^α, so the measure, υ_Φ, is the convolution of the measure associated with these one dimensional representations:

(2.6) $$\upsilon_\Phi = (\iota_{\alpha_1})_* \, dt * \ldots * (\iota_{\alpha_n})_* dt$$

As we'll see below, it is not difficult to compute the RHS of (2.6) by induction on N.

Another useful formula for the measure, υ_Φ, is the following: Consider Φ as the composition of the mapping:

$$\Psi : V \to \mathbf{R}^N, \ (z_1, \ldots, z_N) \to (|z_1|^2, \ldots, |z_N|^2)$$

and the linear mapping

$$L : \mathbf{R}^N \to (\mathbf{R}^n)^*, \quad (s_1, \ldots, s_N) \to s_1\alpha_1 + \ldots + s_N\alpha_N$$

LEMMA 2.3. *Let χ be the characteristic function of the «N-tant» $\{s_1 \geq 0, \ldots, s_N \geq 0\}$, in $\mathbf{R}^N$. Then*

(2.7) $$\Psi * \left(\frac{1}{\pi i}\right)^N dz \wedge d\bar{z} = \chi d s_1 \wedge \ldots \wedge d s_N$$

Proof. This is just formula (2.6) with $N = n$ and $\alpha_1, \ldots, \alpha_N$ the standard basis vectors in $\mathbf{R}^N$. Q.E.D. ∎

Thus we obtain for υ_Φ the formula

(2.8) $$\upsilon_\Phi = L_*(\chi d s_1 \wedge \ldots \wedge d s_N)$$

From the RHS of this formula one immediately deduces:

PROPOSITION 2.4. *If the α_i's span $(\mathbf{R}^n)^*$, then υ_Φ is absolutely continuous with respect to Lebesgue measure, i.e. it can be written in the form*

(2.9) $$\upsilon_\Phi = f(x) \, dx$$

$f(x)$ being a locally $\mathcal{L}^1$ function.

Proof. If the α_i's span, one can make a linear change of coordinates in $\mathbb{R}^N$ so that L becomes the mapping: $x_i = s_i$, $i = 1, \dots, n$; and in these coordinates the proposition is just a consequence of Fubini's theorem. Q.E.D. ∎

The value of f at any point in $(\mathbb{R}^n)^*$ is proportional to the volume of the intersection of $L^{-1}(\alpha)$ with the N-tant : i.e. $f(\alpha)$ *is equal to a constant (not depending on* α *) times the volume of the set*

$$\alpha = \sum s_1\alpha_1 + \dots + s_N\alpha_N, \; s_1, \dots, s_n \geq 0, \tag{2.10}$$

Notice that if α and the α_i's are large, and α is an integer lattice point, this volume is, to a first approximation, proportional to the number of integer lattice points lying in this set, i.e. $f(\alpha) \approx \gamma N(\alpha)$, where N is the *partition function* associated with the weights, $\alpha_1, \dots, \alpha_N$.

The third description of v_Φ is as the fundamental solution of a partial differential equation. Suppose $\alpha_i = (a_{i1}, \dots, a_{in})$. Let

$$\mathrm{D}_{\alpha_i} = a_{i1}\partial/\partial x_1 + \dots + a_{in}\partial/\partial x_n,$$

and let δ_0 be the Dirac delta function with support at the origin in $(\mathbb{R}^n)^*$. We will show that

$$\mathrm{D}_{\alpha_i} \dots \mathrm{D}_{\alpha_N} v_\Phi = \delta_0 . \tag{2.11}$$

Proof. Suppose first that $N = 1$ and $\alpha_1 = \alpha$. Evaluating the RHS on a smooth, compactly supported function, f, we get

$$\begin{aligned}(\mathrm{D}_\alpha v_\Phi, f) &= -(v_\Phi, \mathrm{D}_\alpha f) = \\ &= -((\iota_\alpha)_* \mathrm{d}t, \mathrm{D}_\alpha f) \\ &= \int_0^\infty \sum_1^n a_i \frac{\partial f}{\partial x_i}(a_1 t, \dots, a_n t)\,\mathrm{d}t \\ &= \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t} f(a_1 t, \dots, a_n t)\,\mathrm{d}t = f(0),\end{aligned}$$

which is just δ_0 evaluated on f. To prove (2.11) in general we go back to the formula (2.6). This gives us for the right hand side of (2.11) the expression

$$(\mathrm{D}_{\alpha_1}(\iota_{\alpha_1})_* \mathrm{d}t) * \dots * (\mathrm{D}_{\alpha_N}(\iota_{\alpha_N})_* \mathrm{d}t).$$

As we've just seen, each factor in this convolution product is the delta function supported at the origin; hence so is the product itself. Q.E.D. ∎

By assumption the cone (2.2) is properly contained in some half-space

$$(H_\xi)^+ = \{x \in (\mathbf{R}^n)^*,\ (\xi, x) > 0\}.$$

Hence the measure, υ_Φ, is properly supported in this half-space. We claim that this property and the equation, (2.11), *completely characterize* υ_Φ. Indeed, suppose we were given two distributions both of which satisfied (2.11) and were supported in this half-space. Then their difference, υ, would satisfy

$$\mathrm{D}_{\alpha_1} \dots \mathrm{D}_{\alpha_N} \upsilon = 0$$

and would also be supported in this half-space. Let

$$\upsilon' = \mathrm{D}_{\alpha_2} \dots \mathrm{D}_{\alpha_N} \upsilon$$

Then $\mathrm{D}_{\alpha 1}\ \upsilon' = 0$, and so the support condition clearly implies that $\upsilon' = 0$. Similarly $\mathrm{D}_{\alpha i} \dots \mathrm{D}_{\alpha_N} \upsilon = 0$ for all i, and, in particular, υ itself has to be equal to zero. Notice, by the way, that if we differentiate υ_Φ by just one of the $\mathrm{D}_{\alpha i}$'s, says $\mathrm{D}_{\alpha 1}$, we get, by the same argument as above,

$$\mathrm{D}_{\alpha_1} \upsilon_\Phi = (\iota_{\alpha_2})_* \mathrm{d}t * \dots * (\iota_{\alpha_N})_* \mathrm{d}t. \tag{2.12}$$

We will make use of this identity below.

Next let us investigate some properties of the measure υ_Φ. For every subset, S, of the set of weights, $\{\alpha_1, \dots, \alpha_N\}$, let

$$(V_O)^S = \prod_{\alpha \in S} (V^\alpha - 0)$$

It is easy to see that at every point, p, of $(V_O)^S$, the subgroup of G which stabilizes p is independent of p and is the group

$$\{x \in \mathbf{R}^n/\mathbb{Z}^n, \exp 2\pi i(\alpha, x) = 1, \alpha \in S\}. \tag{2.13}$$

Since a point of V is a critical point of Φ iff its stabilizer group is not discrete, we conclude;

LEMMA 2.5. *The set of critical points of Φ is a disjoint union of $(V_O)^S$'s; moreover, a $(V_O)^S$ is critical iff the α's in S are not a set of spanning vectors of $(\mathbf{R}^n)^*$.* ∎

Let W^S be the following subset of $(\mathbf{R}^n)^*$;

$$W^S = \{\sum s_\alpha \alpha, \alpha \in S, s_\alpha \geq 0\}.$$

From the lemma and the definition of Φ, we get:

PROPOSITION 2.6. *The critical values of* Φ *are the union of the* W^S*'s for which* S *is not a spanning set of* $(\mathbf{R}^n)^*$.

Let Δ_0 be the complement in Δ of the set of critical values of Φ, and let $\Delta_i, i = 1, \ldots, r$, be the connected components of Δ. By proposition 2.6, the Δ_i's are open conic polytopes, and the sets, W^S are the walls of these polytopes. Now lets write υ_Φ as the product of Lebesgue measure with a locally $\mathcal{L}^1$ summable function, f, as in (2.9). Since each Δ_i is contained in the set of regular values of Φ, the restriction of f to Δ_i is a smooth function. Notice also that, by (2.8), the measure, υ_Φ, is the push-forward by a linear map of a measure on $\mathbf{R}^N$ which is homogeneous of degree N with respect the group of homotheties of $\mathbf{R}^N$. Thus υ_Φ is also homogeneous of degree N, and so, by (2.8), f is homogeneous of degree $N - n$, i.e.

$$f(tx) = t^{N-n} f(x).$$

We will now prove the Duistermaat-Heckman theorem in this linear setting:

THEOREM 2.7. *The restriction of* f *to each* Δ_i *is a homogeneous polynomial of degree* $N - n$.

Proof. Choose coordinates in $(\mathbf{R}^N)^*$ so that the Δ is properly contained in the half-space, $x_1 \geq 0$, and α_1 is the unit vector pointing in the direction of the positive x_1 axis. By (2.12)

$$\partial/\partial x_1 \upsilon_\Phi = (\iota_{\alpha_2})_* dt * \ldots * (\iota_{\alpha_N})_* dt. \tag{2.14}$$

Assume by induction that the RHS is a sum of the form

$$\sum g_i(x_1, \ldots, x_n)\chi_i$$

where the g_i's are polynomials and the χ_i's are the characteristic functions of the Δ_i's. Let p be a point in Δ_0 and let $x_1, \ldots, x_n$ be its coordinates. By integrating (2.14), we get

$$f(x_1, \ldots, x_n) = \sum \int_0^{x_1} \chi_i g_i(s, x_2, \ldots, x_n) ds \tag{2.15}$$

Since p is in Δ_0, the ray, $p + t\alpha_i, -\infty < t < 0$, intersects the $n-1$ dimensional walls of the Δ_i's transversally and doesn't intersect any of the lower dimensional walls. Thus, since each Δ_i is convex, one of the three following alternatives has to be true: 1. The ray doesn't intersect the boundary of Δ_i at all.

2. It intersects the boundary of Δ_i in just one point, (in which case p is an interior point of Δ_i.)

3. It intersects the boundary of Δ_i in two points.

Moreover, in the last two cases, the points of intersection depend linearly on p : i.e. in the second case, the point of intersection,

$$p' = (x', x_2, \ldots, x_n)$$

satisfies a linear equation

$$x' = a_1 x_1 + \ldots + a_n x_n,$$

and in the third case the points of intersection,

$$p' = (x', x_2, \ldots, x_n)$$

and

$$p'' = (x'', x_2, \ldots, x_n)$$

satisfy linear equations

$$x' = a_1 x_1 + \ldots + a_n x_n$$

and

$$x'' = b_1 x_1 + \ldots + b_n x_n.$$

In the first case, the ith term makes no contribution at all to the sum (2.15). In the second case, it makes the contribution

$$\int_{x'}^{x_1} f_i(s, x_2, \ldots, x_n) \, \mathrm{d}s$$

and, in the third case, the contribution

$$\int_{x'}^{x'''} f_i(s, x_2, \ldots, x_n) \, \mathrm{d}s$$

It is clear in either case that this expression is a polynomial function of the x_i's. Q.E.D. ∎

REMARK. This proof can be converted into a fairly efficient algorithm for computing f.

Next we will derive a formula for the «jumps» in f across walls separating two adjacent Δ_i's. (We will take pains, by the way, to write this formula as «intrinsically» as possible, because, as we will see in section 5, the version of this formula that we will give below is true in the manifold setting as well.) Let $W = W^S$ be an $n-1$ dimensional wall separating the regions, Δ_+ and Δ_-, and let G^S be the subgroup of G defined by the set of equations (2.13). Since the α's belonging to S span an $n-1$ dimensional subspace of $(\mathbf{R}^n)^*$, this group in one dimensional. Let ξ be a non-zero element in its Lie algebra. Its clear that $\langle\alpha,\xi\rangle = 0$ for all $\alpha \in S$, and that these equations determine ξ up to a constant multiple. Conversely we can assume that S consists exactly of those weights for which $\langle\alpha,\xi\rangle = 0$. We will fix the orientation of ξ by requiring that it be the outward normal to the region, Δ_-. With the convention, ξ is determined up to a positive constant multiple.

Let V^S be the subspace of V spanned by the V^α's in the sum (2.1) with $\alpha \in S$. By (2.13), V^S is the fixed point set of the group, G^S : so, by restricting ρ to V^S, we get a representation of the quotient group, G/G^S, on V^S. Its moment map is just the restriction of Φ to V^S, and maps V^S onto the $n-1$ dimensional wall, W^S. We will denote by υ_S the analogue of the measure, υ_Φ, for the action of G/G^S on V^S and think of this measure as living on W^S. We will show below that *this measure is all the data that is needed to compute the jump in f across W^S.*

Just as for υ_Φ we can write υ_S as the product of a locally $\mathcal{L}^1$ function, f_S, (defined on W^S), times the Lebesgue measure on W^S. A slight hitch is that the Lebesgue measure on W^S is only defined up to multiplication by a positive constant. However, the choice of ξ gives us a way of fixing this constant. Let $(\xi_1,\dots,\xi_n)$ be the coordinates of ξ, and let ν_S be an $n-1$ form on $(\mathbf{R}^n)^*$ of the form

$$\nu_S = \sum_{i=1}^{n} a_i(-1)^i \mathrm{d}x_1 \wedge \dots \wedge \widehat{\mathrm{d}x_i} \wedge \dots \wedge \mathrm{d}x_n$$

the a_i's being constants which satisfy

$$\sum a_i\xi_i = 1. \tag{2.16}$$

Then the restriction of ν_S to W^S doesn't depend on the choices of the a_i's and defines both an orientation and a measure on W^S. This measure, which we will continue to denote by ν_S, will be by definition, our Lebesgue measure on W^S. In terms of it we can write

$$\upsilon_S = f_S\nu_S. \tag{2.17}$$

Notice, by the way, that f_S is itself, (by theorem 2.7 applied to the action of G/G^S on V^S), a piecewise polynomial function. Also notice that if we multiply ξ by a positive number, λ, then by (2.16) ν_S gets multiplied by a factor of $1/\lambda$ and hence f_S gets multiplied by a factor of λ.

Before stating our main result we need two final pieces of notation. Via the identification, $\mathbf{R}^n \cong (\mathbf{R}^n)^{**}$, we can think of ξ as a linear functional on $(\mathbf{R}^n)^*$. It will cause untold confusion below if we use the same notation for ξ and for this linear functional, so we will denote this linear functional by L_ξ. A second bit of notation that we will need is the following. In the decomposition, (2.1), we can assume that the α_i's are so labelled that the first m α_i's are not in S and the remaining α_i's are.

THEOREM 2.8. *Suppose f is equal to the polynomial, f_+, on Δ_+ and f_- on Δ_-. Then*

$$f_+ - f_- = \left(\frac{1}{(m-1)!} \prod_{i=1}^{m} \langle \alpha_i, \xi \rangle^{-1} \right) f_S \, L_\xi^{m-1} + g \tag{2.18}$$

g being a polynomial which vanishes to order m on W^S.

Proof. (By induction on m.) Assume by induction that along W^S the distribution

$$(\iota_{\alpha_2})_* dt * \ldots * (\iota_{\alpha_N})_* dt$$

has a singularity of the form

$$\left(\frac{1}{(m-2)!} \prod_{i=2}^{m} \langle \alpha_i, \xi \rangle^{-1} \right) f_S \, L_\xi^{m-2}$$

By (2.12) this distribution is the derivative of f with respect to D_{α_1}. However,

$$L_\xi^{m-2} = \frac{\langle \alpha_1, \xi \rangle^{-1}}{m-1} \mathrm{D}_{\alpha_1} L_\xi^{m-1}$$

so f itself has to have a singularity of the form, (2.18), along W^S. Q.E.D. ■

REMARK. Notice that the number of weights in S is at least $n-1$; so m is less than or equal to $N-n+1$. The formula, (2.18), is particularly simple when the this inequality is an equality, i.e. when $m = N - n + 1$. Then, since f_+ and f_- are homogeneous polynomials of degree $N-n$, g has to be zero and f_S has to be a constant. This constant is easy to compute: Let

$$\begin{aligned} \alpha_{m+1} &= (a_{1,1}, \ldots, a_{1,n}) \\ &\vdots \\ \alpha_N &= (a_{n-1,1}, \ldots, a_{n-1,n}) \end{aligned} \tag{2.19}$$

and let A_ξ be the matrix having the vectors (2.19) as its first $n-1$ rows and $(a_1, \dots, a_n)$ as its last row. (the a_i's being as in (2.17).) Then

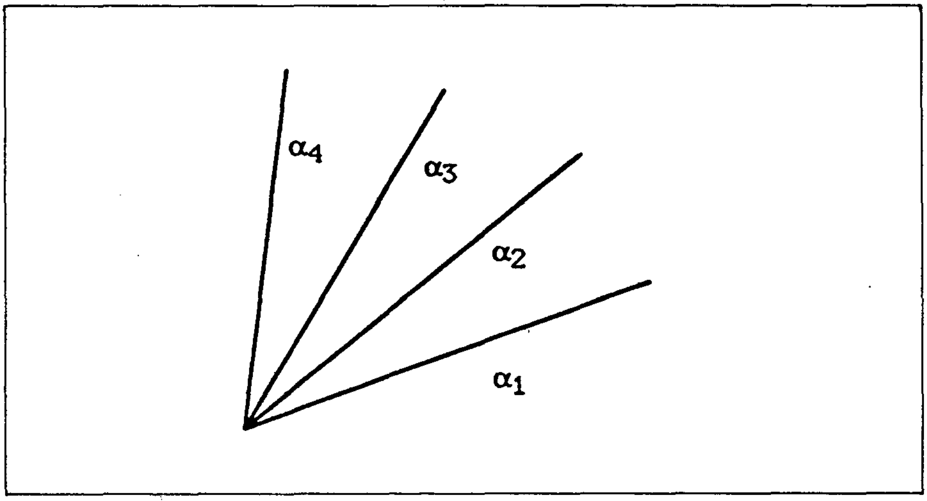
$$f_S = c_S = \det (A_\xi)^{-1} \tag{2.20}$$

and (2.18) reduces to

$$f_+ - f_- = \left(\frac{c_S}{(N-n)!} \prod_{i=1}^{N-n+1} \langle \alpha_i, \xi \rangle^{-1} \right) (L_\xi)^{N-n} \tag{2.18}$$

i.e. the jump across W^S is a *constant* multiple of the monomial, $(L_\xi)^{N-n}$.

EXAMPLE. $N = 4, n = 2, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ distinct. One gets the figure below for Δ :



In all three regions of this figure the D-H polynomials are homogeneous quadratic polynomials. In the exterior regions these polynomials are monomials whose level sets are straight lines parallel to the exterior sides. If one goes into the interior region from the side parallel to α_2, the jump term which one has to add is a monomial whose level sets are straight lines parallel to α_2. If one goes into the interior region from the side parallel to α_3, the jump term which one has to add is a monomial whose level sets are straight lines parallel to α_3. (Exercise: Determine the coefficients of these jump terms using the fact that one has to get the same answer, whether one goes into the region from the left or the right.) One important consequence of (2.18) is the following:

THEOREM 2.9. *f is continuous near W^S if $m > 1$, and is k times differentialble near W^S if $m > k+1$.* ∎

SECTION 3. COMPUTING THE $D-H$ POLYNOMIALS: THE «HECKMAN» FORMULA

Lets now come back to the situation we were considering in section 1: G an n-torus, M a compact symplectic manifold and

$$\kappa : G \to \text{Symplectico}\,(M)$$

a Hamiltonian action of G on M. As in section one, we will denote by

$$\Phi : M \to g^*$$

the moment map associated with κ and by v_Φ the push-forward with respect to Φ of the canonical symplectic measure on M. Let f be the Duistermaat-Heckman function, i.e. the Radon-Nikodym derivative of v_Φ with respect to Lebesgue measure on g^*. The goal of this section will be to derive a formula for f similar to the Kostant-Heckman formula which we discussed in the first paragraph of section 1. Unfortunately, this formula will only make sense when the fixed point set of is a finite set; so we will henceforth assume that this is the case. Before stating this formula in its full generality, we will first describe what a «piece» of this formula looks like. As we pointed out in section one, the support, Δ, of v_Φ is the convex hull of the image of the fixed point set, M_G, of G; so, in particular, Δ is a finite polytope, and its vertices are images of points in M_G. However, there may be points in M_G which don't correspond to vertices of Δ : it is possible for the image of some fixed point to be contained in the interior of the convex hull of the remaining fixed points. In fact, as mentioned in the introduction, a necessary and sufficient condition for a fixed point, p, to correspond to a vertex of Δ is the following: Let τ_p be the linear isotropy representation of G on the tangent space to M at p. By the equivariant Darboux theorem there exists a G invariant neighbourhood, U_0, of the origin in TM_p, and a symplectomorphism:

$$h : (U,p) \to (U_0,0) \tag{3.1}$$

intertwining κ and τ_p. (For a proof of this «G-equivariant Darboux theorem» see [G-S3].) Now let

$$\alpha_{p,i}, \quad i = 1,\dots,N, \tag{3.2}$$

be the weights of the representation of G on TM_p, and let $z_1,\dots,z_N$ be a system of complex coordinates in TM_p compatible with the decomposition of TM_p into weight spaces. By (3.1) the moment map, Φ, restricted to U, is equal to $\Phi_0 \circ h + \Phi(p)$, where

$$\Phi_0(z_1,\dots,z_N) = \sum \alpha_{p,i}|z_i|^2 . \tag{3.3}$$

(Compare with (2.2).) Thus the image of U in g^* is the intersection of a neighborhood of $\Phi(p)$ with the cone:

(3.4) $$\{\Phi(p) + \sum s_i \alpha_{p,i},\ s_1 \geq 0, \ldots, s_N \geq 0.\}$$

and so, $\Phi(p)$ will be a vertex of Δ, at least locally, iff (3.4) is a *proper* cone, i.e. iff there exists a $\xi \in g$ such that

(3.5) $$\langle \alpha_{p,i}, \xi \rangle > 0$$

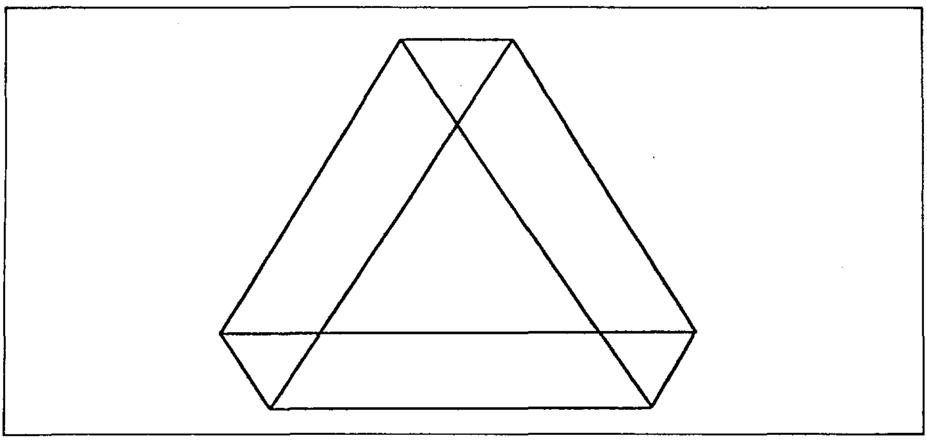
for all i. One can, in fact, prove a good deal more. Using some global properties of the moment mapping, one can show that if $\Phi(U)$ is contained in the cone (3.4), then $\Phi(M)$ is contained in this cone; so if (3.5) holds, $\Phi(p)$ is an honest-to-god vertex of Δ, not just a vertex locally, (See [A] or [GS,].)

Lets now assume that a ξ satisfying (3.5) exists. Then, by lemma 2.2, $\Phi : U \to g^*$ is proper; so the measure, v_Φ, is identical in a neighborhood of $\Phi(p)$ to the measure which we studied in the previous section. In particular:

THEOREM 3.1. *In a neighborhood of* $\Phi(p)$, *the Duistermaat-Heckman function,* $f(\mu)$, *is equal to a fixed constant times* $f_p(\mu + \Phi(p))$ *where*

(3.6) $$f_p(\mu) = \text{volume } \{\mu = \sum s_i \alpha_{p,i},\ s_1 \geq 0, s_N \geq 0.\}$$

Notice, by the way, that since f is a polynomial on each of the subregions, Δ_i, of Δ, one can take the neighborhood on which $f(\mu)$ is proportional to $f_p(\mu + \Phi(p))$ to be a lot larger than $\Phi(U)$: one can take it to be the union of all the Δ_i's whose closures contain $\Phi(p)$. For example, Let M be a six dimensional coadjoint orbit of $SU(3)$ and G the maximal torus of $SU(3)$. The image of the moment map is then a hexagon, and in general, this hexagon will be subdivided into seven regions as depicted below:



(There will be a one parameter family of six dimensional orbits (corresponding to multiples of the adjoint representation) where the hexagon will be a regular hexagon and the middle region will have degenerated to a point, leaving six subregions instead of seven.) In the generic case each vertex will give rise to two subregions, a triangular and a rhomboidal subregion on which f is proportional to (3.6).

The moral of the discussion above is that one can completely describe the measure, v_Φ, in the vicinity of the vertices of Δ using nothing more than the relatively elementary results of the previous section and the G-equivariant Darboux theorem. However, to get information about v_Φ in regions, Δ_i, not containing vertices in their closures, we will need one of the deeper results of the Duistermaat-Heckman theory: the *exact stationary phase formula.* We will give a careful description of this formula below ; but before we do so, lets first recall what the lemma of stationary phase in its usual form is about: Let M be a compact n dimensional manifold, μ a smooth, non-vanishing measure on M, and $\psi : M \to \mathbb{R}$, a smooth function having only a finite number of critical points, all of them non-degenerate. Then the lemma of stationary phase is a recipe for evaluating the oscillatory integral

$$\int e^{i\lambda\psi} d\mu$$

in terms of data at the critical points. More explicitly, it says that for λ large, this integral is equal to

$$\lambda^{-n/2} \sum c_p e^{i\pi/4 \; sgn(p)} \exp i\,\psi(p) \tag{3.7}$$

modulo an error term of order, $O(\lambda^{-n/2-1})$, the sum taken over the fixed points. The c_p's and $sgn(p)$'s in this sum are defined as follows: The Hessian of Ψ at p is a non-degenerate quadratic form on the tangent space to M at p; so one can choose a basis, $v_1, \dots, v_n$, for the tangent space such that with respect to this basis

$$D^2\psi(v_i, v_j) = \epsilon_i \delta_{ij},$$

where $\epsilon_i = \pm 1$. The number of $+1$'s is $sgn(p)$, (the signature of $D^2\psi$), and c_p is the quantity, $\mu(v_1, \dots, v_n)$. In particular, let M be the symplectic manifold that we've been considering, μ the canonical symplectic measure, and ψ a component of the moment mapping: i.e. let ξ be an element of g, and let $\psi = \phi^\xi = (\Phi, \xi)$. It is easy to see that ϕ^ξ has nondegenerate critical points if and only if the conditions (1.6) are satisfied for all fixed points, p, of G; and, if these conditions are satisfied, the critical points of ϕ^ξ coincide with these fixed points, in fact, by (3.3) the Hessian of ϕ^ξ at p is

$$\sum \alpha_{p,i}(\xi)|z_i|^2, \tag{3.8}$$

and its clear that this quadratic form is non-degenerate if and only if the conditions, (3.5), are satisfied. From (3.3) one computes for the contribution of p to (3.7)

$$c_p e^{(\pi/2) i\, sgn(p)} = (i)^N (\prod \alpha_{p,i}(\xi))^{-1}$$

where $N = \dim M/2$. Thus the modulo an error term stationary phase formula for ϕ^ξ reduces to

$$\int e^{i\lambda(\Phi,\xi)} d\mu = (-i\lambda)^{-N} \sum \exp i(\Phi(p),\xi)/(\prod(\alpha_{p,i}(\xi)) \tag{3.9}$$

for all ξ which satisfy (1.6). The exact stationary phase formula of Duistermaat-Heckman says that the error term is *identically zero.* In other words, setting $\lambda = 1$, the identity

$$\int e^{i(\Phi,\xi)} d\mu = (i)^N \sum \exp i(\Phi(p),\xi)/(\prod(\alpha_{p,i}(\xi)) \tag{3.10}$$

holds on the nose. For the proof we refer to [D-H] or [B-V] or [G-S3]. For the moment we will just say a few words about how one can «almost» prove (3.10) just using the G-equivariant form of the Darboux theorem. Namely, by the Darboux theorem, one can write the integrand on the left hand side as

$$\exp(i\lambda \sum \alpha_{p,i}(\xi)|z_i|^2)\ (1/\pi i)^N\, dz \wedge d\bar{z},$$

in the neighborhood of the fixed point. Thus, by a partition of unity argument, one can write (3.9) as a sum of the integrals

$$\int \exp(i\lambda \sum \alpha_{p,i}(\xi)|z_i|^2)\ (1/\pi i)^N\, dz \wedge d\bar{z}, \tag{3.11}$$

(one such integral for each fixed point), and an integral of the form

$$\int \rho \exp(i\lambda\phi^\xi) d\mu$$

where ρ is identically zero in the neighborhood of the critical set. By elementary Fourier analysis, the second expression is of order, $O(\lambda^{-\infty})$ in λ for λ large; and, by elementary calculations, one can show that the sum of the terms, (3.11), is equal to the first term on the right hand side in (3.9). Thus it takes no effort at all to improve the $O(\lambda^{-N/2-1})$ in (3.9) to an $O(\lambda^{-\infty})$. The hard part of the proof of (3.10) is getting rid of this innocuous looking $O(\lambda^{-\infty}.)$

We will now describe the «Heckman» formula, (1.23), alluded to at the beginning of this section. Consider the subset of g consisting of the union of the hyperplanes

$$(\alpha_{p,i},\ \xi) = 0.$$

The component of this set has several connected components, and we will fix, once and for all, one of these components and call it our *positive Weyl chamber.* (Formula (1.23) which we derive below will actually be a collection of formulas, one for each choice of a positive Weyl chamber.)

As described in the introduction, we define, for each fixed point, p, a *renormalized* set of weights by equation (1.7) and define the υ_p by (1.21) and (1.22), and the function $(-1)^p$ by (1.10).

From the previous section we known that υ_p is a constant multiple of $f_p \mathrm{d}\,x$, f_p being the «Heckman partition function»

$$f_p(\mu + \Phi(p)) = \text{volume } \{\mu = \sum s_i \alpha^w_{p,i}, s_1 \geq 0, \dots, s_N \geq 0\}$$

THEOREM. *The Duistermaat-Heckman measure,* υ_Φ, *is equal to the alternating sum over the fixed point set of* G :

$$\upsilon_\Phi = \sum (-1)^p \upsilon_p .$$

Proof. For the moment fix a point, p, in the fixed point set, and set

$$\alpha_i = \alpha^w_{p,i}, i = 1, \dots, N$$

and

$$v = \Phi(p) .$$

Consider the function

$$g(\xi) = \frac{e^{i(v,\xi)}}{\prod_{i=1}^{N} (\alpha_i, \xi)} . \tag{3.12}$$

Let O be the open subset of g obtained by deleting the hyperplanes, $(\alpha_i, \xi) = 0$. Then g is a well-defined rational function on O, and there are many ways to extend it to a «generalized» function on all of g. ■

PROPOSITION 3.3. *Let* ξ *be a vector in the positive Weyl chamber. Then there is one and just one tempered distribution,* h, *on* g *with the properties that* h *is equal to* g *on* O *and that the Fourier transform of* h *is properly supported in the half-space,*

$$\{\alpha \in g^*, \langle \alpha, \xi \rangle \geq c\} \tag{3.13}$$

for some constant, c.

Proof. Without loss of generality we can assume that $v = 0$, (in which case we can take the constant, c, to be zero). The existence of one such distribution is easy. Let υ be the measure defined by (2.4). We showed that this measure satisfies the differential equation

$$D_{\alpha_1} \dots D_{\alpha_N} \upsilon = \delta_0 ;$$

so the Fourier transform of this measure satisfies

$$\left(\prod (\alpha_i, \xi)\right) \hat{\upsilon} = 1 ,$$

which reduces to (3.12) on O. As for the uniqueness, suppose there are two distributions with the properties above. Let k be their difference. Then k is supported on the union of the hyperplanes, $(\alpha_i, \xi) = 0$. Moreover, since it is tempered, it lies in some fixed Sobolev space, and hence (regarded as a linear functional on the Schwartz space) is continuous in the C_0^r topology for a sufficiently large r. Therefore, its product by a sufficiently large power of $\prod \langle \alpha_i, \xi \rangle$ has to vanish, and its Fourier transform has to satisfy the differential equation,

$$(3.14) \qquad (D_{\alpha_1} \dots D_{\alpha_N})^r f = 0 ,$$

for r large. However, since f is supported on a half-space of the form

$$(3.15) \qquad \{\mu \in g^*, \langle \mu, \xi \rangle \geq c\},$$

with ξ in the positive Weyl chamber, this implies that f is identically zero. (See the discussion in section 2 following the proof of (2.11).) Q.E.D. ∎

To prove (1.23) notice that, by construction, the Fourier transform on the LHS of (1.23) is equal to the Fourier transform of the RHS on O. However, by the Fourier inversion formula, the Fourier transform of these Fourier transform are both supported in a half-space of the form, (3.15). Hence a repetition of the previous argument shows that they have to be identical. Q.E.D.

SECTION 4. IS THERE A «KOSTANT» FORMULA CORRESPONDING TO THE «HECKMAN» FORMULA?

We will show in this section that the answer to this question is yes and is given by (1.13). We will have to assume, however, that the symplectic action of G on M, i.e.

$$\kappa : G \to \text{Symplectico } (M)$$

can be «quantized». There is no firm consensus about what this term means; however, for the duration of section 4, we will take it to mean that the following are true.

a. κ can be *pre-quantized*: Suppose that the cohomology class defined by the symplectic form on M belongs to the image of the canonical mapping, $\iota : H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})$, i.e. that it is an *integral* cohomology class. Then, by [K], there exists a Hermitian line bundle, L, sitting over M and a connection, ∇, on L whose curvature form is ω_M. Moreover, if M is simply connected, L and ∇ are unique up to isomorphism. Furthermore, (loc. cit.), one gets a canonical representation of g on the space of sections of L : For each $\xi \in g$, let $\xi^\#$ be the vector field on M associated with ξ, and let

$$\mathrm{D}_\xi = \nabla_{\xi^\#} + i\langle \Phi, \xi \rangle. \tag{4.1}$$

Then the map, $\xi \to \mathrm{D}_\xi$, is a Lie algebra homomorphism. Following [K] we say that κ can be pre-quantized if there is a representation of G on sections of L which is given infinitesmally by (4.1).

b. M possesses a positive definite G-invariant polarization, i.e. there exists a G invariant Kaehler structure on M compatible with its symplectic structure. In this case L aquires naturally the structure of a holomorphic line bundle, and the holomorphic structure on L is compatible with the action of G defined by (4.1).

Given a. and b. we will, again following [K], define the quantization of κ to be the natural representation, ρ, of G on the space of holomorphic sections of L. Our starting point for the «generalized Kostant formula» (1.13) below will be a formula for the «character» of ρ (actually an alternating trace, i.e. a Leftschetz number) which is very similar to the exact stationary phase formula described in the previous section. As in the previous section, we will assume that the fixed point set, M_G, is finite and that for each $p \in M_G$, the weights of the isotropy representation of G on the tangent space to M at p are given by (3.2). Let ξ be an element of g satisfying (3.5), and consider the Lefschetz number $L(\rho(\exp \xi))$. By the Atiyah-Bott fixed point formula, [A-B], this is equal to the sum over M_G of

$$\sum \frac{tr\kappa_p(\exp \xi) : L_p \to L_p}{\prod(1 - \exp i\langle \alpha_{p,\kappa}, \xi \rangle}$$

κ_p being the isotropy representation of G on the fiber, L_p. By (4.1) the trace of $\kappa_p(\exp \xi)$ is equal to $\exp i(\Phi(p), \xi)$; so we get formula (1.4).

Before exploiting this formula, we will pause for a moment to point out some similarities between it and (3.10). In both formulas, the sum is over the fixed point set of G and the numerator of the $p-th$ summand in the same. Moreover, for ξ small, the denominator of the $p-th$ summand in (1.4) is approximately equal to

$$(-i)^N \prod \alpha_{p,k}(\xi) \tag{4.2}$$

which is identical with the denominator of the $p-th$ summand in (3.10). Thus we've proved

PROPOSITION 4.1. For ξ small, the trace of $\rho(\exp i\xi)$ is approximately equal to $\hat{\upsilon}_{\Phi}(\xi)$.

This result explains, incidentally, why the asymptotic properties of the multiplicity diagrams can be described by a «central limit theorem» involving the measure, υ_{Φ}; for, by rescaling, «ξ small» in proposition 4.1 can be reinterpreted as «the $\alpha_{p,i}$'s large».

Lets now consider the Fourier transforms of the right and left hand sides of (1.4). Let α be a lattice point in g^* and let $\ell(\alpha, L)$ be the Lefschetz number of α in ρ. Then the LHS of (1.4) is the sum

$$\sum \ell(\mu, L)\exp i(\mu, \xi)$$

and its Fourier transform is a finite sum of delta functions:

$$\sum \ell(\mu, L)\delta(\lambda - \mu). \tag{4.3}$$

On the other hand to compute the Fourier transform of the RHS of (1.4), we have to know how to compute the Fourier transforms of each of the individual terms on the RHS. The situation here is similar to the situation we encountered in the previous section in trying to compute the RHS of (3.10): there is some ambiguity about the definition of this Fourier transform since the $p-th$ summand in (4.2) is defined only if

$$\langle\alpha_{p,k}, \xi\rangle \notin 2\pi\mathbb{Z}$$

for all k and p. As described in section 1, we will deal with this ambiguity by *renormalizing* the $\alpha_{p,k}$'s, i.e. we will fix for once and for all a «positive Weyl chamber» and so choose a ξ satisfying (1.3) and (1.5) for some q, and make the definition (1.7)-(1.12) at each fixed point p. After some juggling we can write the $p-th$ term in the sum of the right of (1.4) as

$$(-1)^{w_p}\frac{\exp i\langle\Phi(p) + \delta_p^w - \delta_p, \xi\rangle}{\prod_{k=1}^{N}\left(1 - \exp i\left(\alpha_{p,k}^w, \xi\right)\right)} \tag{4.4}$$

Now lets replace each factor in the denominator of this expression by the corresponding geometric series

$$\sum \exp im(\alpha_p^w, i, \xi) \tag{4.5}$$

and get for (4.4) the formula

$$(4.6) \qquad (-1)^{p} \sum N_p(\mu) \exp i(\mu + \Phi(p)) + \delta_p^w - \delta_p, \xi)$$

where $N_p(\mu)$ is the number of N-tuples of non-negative integers, $(k_1, \dots, k_N)$, satisfying the equation

$$\mu = \sum k_i \alpha_{p,i}^w .$$

We will can N_p the *partition function* associated with the point, p. Consider now the Fourier transform of (4.6). This is the sum

$$(4.7) \qquad (-1)^{w_p} \sum N_p(\mu) \delta(\lambda - \mu - \Phi(p) - \delta_p^w + \delta_p)$$

Notice that both (4.7) and (4.3) have support in a fixed half-space

$$H_\xi = \{\lambda \in g^*, (\lambda, \xi) \geq c\},$$

ξ lying in the positive Weyl chamber and c being a fixed constant. (For (4.7) this is a consequence of the fact that N_p is supported in the cone,

$$\left\{ \sum s_i \alpha_{p,i}^w, \; s_i \geq 0, \dots, s_N \geq 0 \right\},$$

and for (4.3) it is merely a consequence of the fact that (4.3) is compactly supported.) We will show, as in section 3, that this forces (4.3) to be equal *on the nose* to the sum over $p \in M_G$ of the expression, (4.7). Indeed, take the Fourier transform of (4.3) and of the sum of the (4.7)'s, and let k be their difference. Then, by construction, k is supported on the union of the hyperplanes, $\langle \alpha_{p,i}, \xi \rangle \in 2\pi \mathbb{Z}$, and hence

$$\prod (1 - \exp \langle \alpha_{p,i}^w, \xi \rangle)^r k = 0$$

for sufficiently large r. Therefore, the inverse Fourier transform of k satisfies the difference equation

$$(4.8) \qquad \prod (1 - J_{\alpha_{p,i}^w})^r f = 0,$$

J_α being the operator, «translation by α.» This, together with the fact f is supported in H_ξ forces f to be zero, and hence proves that (4.3) and the sum of the (4.7)'s are equal. (Proof: Argue by induction on the number of factors in the product on the right in the equation (4.8). For instance if r is equal to one and there is just one $\alpha_{p,i}^w$, this equation says that $f(\mu) = f(\mu + \alpha)$. Since f is supported in H_ξ and $\langle \alpha, \xi \rangle \neq 0$, this forces f to be zero.) Lets now compare both sides of this equation. Setting $\upsilon = \mu + \Phi(p) - \Phi(p) + \delta_p^w + \delta_p$, we can rewrite (4.7):

$$(-1)_p^w \sum N_p(\upsilon - \Phi(p) + \delta_p^w - \delta_p) \delta(\lambda - \upsilon).$$

Thus, summing over p and comparing the coefficients of $\delta(\lambda - \upsilon)$, we get the «Kostant» formula (1.13).

SECTION 5. COMPUTING THE $D-H$ POLYNOMIALS: FORMULAS FOR THE JUMPS

In the exact stationary phase formula which we described in section 3.3 we had to assume that the fixed point set of G, i.e. M_G, was a finite set. Duistermaat and Heckman derived in [DH, 2] a more general version of exact stationary phase which makes no assumptions at all about the fixed point set of G. Unfortunately it is not very useful for the computations we are trying to do here, since it involves knowing quite a bit about the topology of the normal bundle of M_G. However, the «leading term» in their formula can be computed by a standard stationary phase argument and is very useful for computing the «jumps» in the $D-H$ polynomials across the walls of the Δ_i's. It turns out that these jumps are given by essentially the same formulas as in the linear case. Here are the details:

Let Δ_+ and Δ_- be two adjacent Δ_i's, let f_+ and f_- be the $D-H$ polynomials associated with them, and let W be the $(n-1)$-dimensional wall separating them, oriented so that the «normal vector», ξ, to W is pointing out of Δ_- and into Δ_+. We will show that the jump, $f_+ - f_-$, at W is given by the formula

$$f_+ - f_- = \left(\prod_{i=1}^{m}\langle\alpha_i,\xi\rangle^{-1}\right) f_W \frac{(L_\xi - L_0)^{m-1}}{(m-1)!} \tag{5.1}$$

plus an error term of order $(L_\xi - L_0)^m$, the quantities, α_i, f_W, L_ξ, etc. being defined more or less as in the linear case. (Compare with (2.18).) The precise definitions are as follows: As we already mentioned, ξ points into Δ_+. (Since W sits in g^*, we can think of ξ as an element of g or as an element of T^*W.) We recall that L_ξ denotes the linear functional on g^* associated with ξ, i.e. the linear functional

$$\mu \to \langle\mu,\xi\rangle.$$

The one-parameter subgroup of G generated by ξ is a circular subgroup, and we will denote it by S^1. To explain the other undefined quantities on the RHS of (5.1), we must first say a few words about the action of S^1 on M. To begin with, its fixed point set, M_{S^1}, is a symplectic submanifold of M, and Φ maps one (or more) of the connected components of M_{S^1} onto W. (For simplicity, we will assume that just one component is involved and denote this component by X.) The quotient group, G/S^1, acts in a Hamiltonian fashion on X, and the moment map of this action is the restriction of Φ to X. The push-forward of the canonical symplectic measure on X by the moment map is a measure which lives on W, and the f_W in (5.1) is the Radon-Nikodym derivative of this measure with respect to the Lebesgue measure on W. (A small complication is: «how to define Lebesgue measure on W.» In principle it is only defined up to the choice of a positive constant; however, as we pointed out in section 2, the choice of ξ fixes this constant in a canonical way. See (2.16) and the discussion following it.)

We still have to explain the «m» and the «α_i's» in (5.1): The integer, m, is as in (2.18), the codimension of X divided by 2, and the α_i's are the weights of the isotropy representation of S^1 on the normal bundle of X. (Notice that, because X is connected, this representation has the same weights at every point.)

Proof of (5.1). Let H be an $(n-1)$-dimensional toral subgroup of G with the property that H and S^1 intersect in $\{e\}$, and let Φ_H be the moment mapping associated with the action of H on X. Φ_H is the composition of Φ with the canonical projection:

$$\pi; g^* \longrightarrow h^*,$$

and the restriction of π to W maps W bijectively onto h^*. Let α be a point on W whose image, $\pi(\alpha)$, is a regular value of Φ_H. Associated to α are two symplectic manifolds: One can reduce M (viewed as an H-space) at $\pi(\alpha)$, and one can reduce X (viewed as a G/S^1-space) at α. Let us denote these reduced spaces by M_α and X_α. Notice that since the action of S^1 commutes with the action of H, there is an induced action of S^1 on M_α.

LEMMA 5.1. *X_α is the fixed point set for the action of S^1 on M_α.* ■

We will leave the proof of this as an easy exercise. Notice that, by definition, the value of f_W at α is the symplectic volume of X_α; so, by the lemma, we are reduced to proving (3.1) in the special case, $G = S^1$; that is, we are reduced to proving the following: (the manifold, M, in the theorem that follows being the $M_\alpha above$.)

THEOREM 5.2. *Let M be a compact symplectic manifold. Suppose S^1 acts on M in a Hamiltonian fashion with moment map, $\phi : M \longrightarrow \mathbf{R}$. Let $f(t)\,\mathrm{d}t$ be the push-forward by ϕ of the symplectic measure on M, and let C_α be the critical set*

$$\{m \in M, \phi(m) = \alpha, \mathrm{d}\,\phi_m = 0\}.$$

Then the jump in $f(t)$ at α is given by the formula:

$$f_+ - f_- = \text{volume } (C_\alpha) \left(\prod_{i=1}^{k} \alpha_i^{-1}\right) \frac{(t-\alpha)^{k-1}}{(k-1)!} \tag{5.2}$$

plus an error term of order $O((t-\alpha)^k)$, the α_i's being the weights of the representation of S^1 on the normal bundle of C_α.

Proof. Let μ be the symplectic volume on M. It is clear that the Fourier transform of f is the integral

$$\int e^{is\phi}\,\mathrm{d}\,\mu. \tag{5.3}$$

The contribution of M_α to the stationary phase expansion of (5.3) is

$$\text{volume } (C_\alpha)\left(\prod_{i=1}^{m}\alpha_i^{-1}\right)s^{-m}e^{is\alpha}+O(s^{-m-1}) \tag{5.4}$$

Here we are just using the standard stationary phase formula for «clean» phase functions. See, for instance, [HO], page 222.) Therefore, by an elementary Tauberian argument, the inverse Fourier transform of (5.4) is equal to (5.3) plus an error term of order

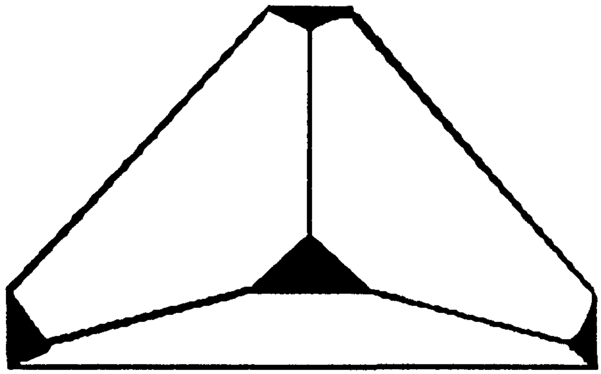
$$O((t-\alpha)^k).$$

Q.E.D. ∎

The formula, (5.1) is particularly useful if the action of G/S^1 on X is a «Delzant» action, i.e. if the dimension of X is as small as possible, namely twice the dimension of the group, G/S^1. In this case, f_W is equal to a constant, (which we will denote by c_W) and (5.1) becomes an exact formula:

$$f_+ - f_- = c_W\left(\prod_{i=1}^{N-n}\langle\alpha_i,\xi\rangle^{-1}\right)\frac{(L_\xi - L_0)^{N-n}}{(N-n)!} \tag{5.5}$$

This formula turns out to very useful for computing the $D-H$ polynomials for Lie groups of low rank. To illustrate how useful (5.5) can be, we will give below a brief sketch of how to compute the $D-H$ polynomials for the ten dimensional coadjoint orbits of $SU(4)$. (The $D-H$ polynomials for these orbits, unlike those for the six and eight dimensional orbits of $SU(4)$ or the six dimensional orbits of $SU(3)$, don't seem to be easy to compute just using the Heckman formula alone.) The coadjoint orbits of $SU(4)$ are just the sets of «isospectral» 4×4 Hermitian matrices. To be more specific, let $\alpha_i, i=1,\dots,4$, be a quadruple of real numbers, normalized so that $\alpha_1\geq\alpha_2\geq\alpha_3\geq\alpha_4$, and $\alpha_1+\dots+\alpha_4=0$; and let $\mathbf{O}_\alpha$ be the set of all 4×4 Hermitian matrices whose eigenvalues are the α_i's. Then every coadjoint orbit of $SU(4)$ is an $\mathbf{O}_\alpha$ and visa versa. The generic coadjoint orbits are those for which all the α_i's are distinct, and these orbits are twelve dimensional. The orbits which we are interested in at the moment are those for which two of the α_i's are equal, and it turns out that these are exactly the ten dimensional ones. Since the group, G, is the diagonal subgroup of $SU(4), g^*\cong g\cong$ the space of 4×4 diagonal matrices of trace zero. The moment map, $\Phi:\mathbf{O}_\alpha\to g$, is just the map which assigns to every matrix belonging to $\mathbf{O}_\alpha$ its diagonal entries. The image, Δ, of the moment map is a «truncated tetrahedron» which looks like



It turns out that the number of Δ_i's is fifteen or forty-nine depending on whether or not $\alpha_2 = \alpha_3$. We will consider the case, $\alpha_2 \neq \alpha_3$, and to be specific, we will assume $\alpha_1 = \alpha_2 = \alpha$; i.e. that the matrices belonging to $\mathbf{O}_\alpha$ have the eigenvalues α, α_3, and α_4. As we mentioned above, the walls of the α_i's are the images, with respect to the moment map, of the fixed point sets of certain circular subgroups of G. It turns out that, up to conjugation, there are only two subgroups that we have to worry about. The first is the group generated by the diagonal matrix with $+1$'s in its first two diagonal entries and -1's in the remaining two entries, and the second the group generated by the diagonal matrix with $+3$ in its first diagonal entry and -1's in the remaining diagonal entries. We will denote the generator of the first group by A and that of the second group by B. Consider, now the action of each of these two groups of the coadjoint orbit, $\mathbf{O}_\alpha$:

PROPOSITION 5.3. *The fixed point set of the group generated by A has four connected components, two $\mathbb{C}P^1$'s and two $\mathbb{C}P^1 \times \mathbb{C}P^1$'s, and for the $\mathbb{C}P^1 \times \mathbb{C}P^1$'s the action of the quotient group, G/S^1, is a Delzant action. The fixed point set of the group generated by B has three components, two of which are $\mathbb{C}P^2$'s; and the action of G/S^1 on these $\mathbb{C}P^2$'s is a Delzant action.*

Proof. The fixed point set of the group generated by A consists of the matrices in the set, $\mathbf{O}_\alpha$, which commute with A; and these are the matrices which have the two by two block form

$$\begin{bmatrix} S & 0 \\ 0 & T \end{bmatrix}$$

Thus the spectrum of S has to be either $\{\alpha\},\{\alpha_3,\alpha_4\},\{\alpha,\alpha_3\}$ or $\{\alpha,\alpha_4\}$; and the corresponding spectrum of T either $\{\alpha_3,\alpha_4\}$, $\{\alpha_3\}$, $\{\alpha,\alpha_4\}$ or $\{\alpha,\alpha_3\}$. Its easy to see that the set of matrices for which the spectrum of S is α or the spectrum of T is $\{\alpha\}$ is a $\mathbb{C}P^1$, and the set of matrices for which the spectrum of S is $\{\alpha,\alpha_3\}$ or the spectrum of T is $\{\alpha,\alpha_3\}$ is a $\mathbb{C}P^1 \times \mathbb{C}P^1$. For B the situation is similar: The matrices belonging to the set $\mathbf{O}_\alpha$ which commute with B have to have the form:

$$\begin{bmatrix} s & 0 \\ 0 & T \end{bmatrix}$$

where $s = \alpha,\alpha_3$, or α_4, and T is a 3×3 Hermitian matrix with spectrum $\{\alpha,\alpha_3,\alpha_3\}$, $\{\alpha,\alpha_4\}$ or $\{\alpha,\alpha_3\}$. In the latter two cases the set of all such matrices is a $\mathbb{C}P^2$. Q.E.D. ∎

Notice from the figure above that Δ has four exterior walls which are hexagonal in shape. From proposition 5.3 it is easy to deduce that these are the only walls of the Δ_i's which are *not* Delzant. In other words

PROPOSITION 5.4. *For the ten dimensional orbits of $SU(4)$, the jumps in the $D-H$ polynomials across walls of the Δ_i's are all of the form (5.5), except for the jumps across the four exterior hexagonal walls.* ∎

In particular, at every interior wall, the change in f is a quadratic monomial whose level sets are planes parallel to the wall.

REFERENCES

[A] M. ATIYAH: *Convexity and commuting Hamiltonians,* Bull. Lond. Math. Soc. 14 (1982), 1-15.

[A-B] M. ATIYAH, R. BOTT: *A Lefchsetz fixed point formula for elliptic complexes I,* Ann. Math. 86 (1967) 374-401.

[B-G] F. A. BEREZIN, I. M. GELFAND: *Some remarks on the theory of spherical functions on symmetric Riemannian manifolds,* Trud. Mos. Mat. O.-va 5 (1956), 311-351.

[B-B-G] I. N. BERNSTEIN, I. M. GELFAND, S. I. GELFAND: *Structure of representations generated by vectors of highest weight* Funct. Anal. Prilozh. 5 (1971) 1-9. I. N. BERNSTEIN, I. M. GELFAND, S. I. GELFAND: *Differential operators on the base space and a study of g-modules* Lie Groups and Their Representations (Summer School of the Bolyai Janos Mathematical Society) Holland Press, New York, 1975, 21-64.

[D-H] J. J. DUISTERMAAT, G. HECKMAN: *On the variation in the cohomology of the symplectic form of the reduced phase space* Inven. Math. 69 (1982) 259-268.

[G-S1] V. GUILLEMIN, S. STERNBERG: *Convexity properties of the moment map,* Invent. Math. 67 (1982) 491-513.

[G-S2] V. GUILLEMIN, S. STERNBERG: *Geometric quantization and the multiplicities of group representations,* Inven. Math. 67 (1982) 515-538.

[G-S3] V. GUILLEMIN, S. STERNBERG: *Symplectic Techniques in Physics*, Cambridge Univ. Press. Cambridge, New York 1982.

[H] G. HECKMAN: *Thesis* Leiden (1982).

[J] N. JACOBSON: *Lie Algebras*, Interscience Pub. New York (1962) 259-262.

[K] B. KOSTANT: *A formula for the multiplicity of a weight*, Trans. Amer. Math. Soc. 93 (1959) 53-73.

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